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GENERAL BIORTHOGONAL MULTI RESOLUTION ANALYSES AND APPLICATIONS

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ABSTRACT: We present in this paper Riesz bases and dual Riesz bases. Next, we define and study general biorthogonal multiresolution analyses on the real line and we prove commutation properties between derivation and projectors. As applications, we prove that the wavelet bases constructed in this paper are adapted for the study of the Sobolev spaces $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})(s \in N)$.

Keywords: Multiresolution analysis, Riesz and Dual basis,Wavelet, Sobolev space.

Mathematics Subject Classication: MSC Code: 42C15; 44A15.

1. Introduction

The search for wavelet bases has been an active eld for many years, since the beginning of the 1990's. Wavelet concepts have unfolded their full computational efficiency mainly in harmonic analysis (for the study of Calderon-Zygmund operators) and in signal analysis. In general, constructions use either the basis of I. Daubechies or the spline basis without theoretical construction. The wavelet expansions induce isomor-phisms between function and sequence spaces. It means that certain Sobolev or Besov norms of functions are equivalent to weighted sequence norms for the coecients in their wavelet expansions. The wavelets have cancelation properties that are usually expressed in terms of vanishing polynomial moments. The combination of the two previous properties of wavelets provides a rigorous analysis of adaptative schemes for elliptic problems. Moreover, nonlinear approximation is an important concept related to adaptative approximation. The multiscale

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bases have been existed for a long time in search of Haar, Franklin and Littlewood-Paley. They are widely used in many scientic domains as numerical analysis or theoretical physics. The scaling and wavelet functions within a biorthogonalization process are generated by locally supported zonal kernel functions. As applications,they are applied successfully to geophysically and geodetically relevant problems involving rotation-invariant pseudodierential operators (Freeden *et Al* (2007)).

We construct in this paper general biorthogonal multiresolution analyses and the associated biorthogonal wavelet bases. The scaling spaces are constructed in an elementary way. The main contribution offered in this paper which diers from the other constructions is the realization of global higher regularity by more elementary techniques than perhaps those involved in (Cohen *et Al* (2000), Dahmen *et al* (2000) and Jouini (2007)). The global regularity is sucient for applications and the bases are easy to implement.

Section 2 is devoted to the description of Riesz bases and dual bases which will be useful for the remainder of the work.

In section 3, we construct and study in the first part the scaling spaces of a general biorthogonal multiresolution. This construction is not complicated and not technical because the scaling spaces are constructed in an elementary way. In the second part, we describe the associated wavelets and we prove the commutation properties between projectors and derivation.

In the last section and as applications, we characterize regular spaces namely Sobolev spaces in terms of discrete norm equivalences.

2. Riesz Bases and Dual Bases

We start this section with the mother wavelet notion, the basic function in wavelet theory. Next, We dene the Riesz basis

Definition 2.1 A family $(e_k)_{k\in\mathbb{Z}}$ is a Riesz basis of a Hilbert space H if there exist two constants $C_2 > C_1 > 0$ such that, for every sequence (α_k) of $\ell^2(\mathbb{Z})$, we have

$$C_1\left(\sum_{k\in\mathbb{Z}}|\alpha_k|^2\right)^{\frac{1}{2}} \leq \left\|\sum_{k\in\mathbb{Z}}\alpha_k e_k\right\| \leq C_2\left(\sum_{k\in\mathbb{Z}}|\alpha_k|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

and the nite linear combinations $\Sigma\alpha_k e_k$ are dense in H.

Definition 2.2 The wavelet function of *J*. Morlet is a function $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that

$$C_\psi = \int_o^{+\infty} |\hat{\psi}(\xi)(t\xi)|^2 \frac{dt}{t} \tag{2.2}$$

where $\hat{\psi}(\xi)$ is the classical Fourier transform of ψ given by:

$$\hat{\psi}(\xi)(\lambda) = \int_{\mathbb{R}} \psi(x)e^{-i\lambda x}dx,$$

and $(0 < C_\psi < +\infty)$ and C_ψ is independent of ξ,

Remark 2.1

i) If $\hat{\psi}(\xi)$ is real, C_ψ is independent of ξ.

In fact by the variable change $(x = t\xi)$, C_ψ becomes

$$\int_o^{+\infty} \left|\hat{\psi}(\xi)(t)\right|^2 \frac{dt}{t}.$$

ii) The conditions of definition 2.2 are satisfied if:

- $\hat{\psi}(\xi)$ is real
- $\hat{\psi}(\xi)$ is continuous at 0
- There exists a constant $\alpha > 0$ such that $\hat{\psi}(\xi) = 0(|\xi|^\alpha)$ near 0.

Proposition 2.1 Let $g(x)$ be a function of $L^2(\mathbb{R})$ such that

$$\int (1 + x^2)\left|g(x)\right|^2 dx > \infty. \tag{2.3}$$

Then, there exists a constant C such that for every sequence $(\alpha_k)_{k\alpha\mathbb{Z}}$ in $L^2(\mathbb{Z})$; $\Sigma_{k\in\mathbb{Z}}\alpha_k g(x-k)$ belongs to $L^2(\mathbb{R})$ and we have

$$\left(\int_{-\infty}^{\infty}\left|\sum_{k\in\mathbb{Z}} \alpha_k g * (x-k)\right|^2 dx\right)^{1/2} \leq C\left(\sum_{k\in\mathbb{Z}} |\alpha_k|^2\right)^{\frac{1}{2}}. \tag{2.4}$$

Definition 2.3 The functions $(g(x-k))_{(k\in\mathbb{Z})}$ are said to be quantitatively independent if there exists a constant $\gamma > 0$ such that

$$\left(\int_{-\infty}^{\infty}\left|\sum_{k\in\mathbb{Z}} \alpha_k g * (x-k)\right|^2 dx\right)^{\frac{1}{2}} \geq \gamma\left(\sum_{k\in\mathbb{Z}} |\alpha_k|^2\right)^{1/2} \tag{2.5}$$

for every sequence (α_k) of scalars.

Lemma 2.1 Let $(g(x-k))_{(k\in\mathbb{Z})}$ be a generating family of a closed subspace V_0 of $L_2(\mathbb{R})$ is a Riesz Basis of V_0 if and only if

$$\left(\sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi + 2k\pi)\right|^2\right)^{1/2} \geq \gamma > 0. \tag{2.6}$$

Proof. Using the Plancherel Formula and the inequality (2.5) , we get the inequality (2.6).

Definition 2.4 The dual basis $(g^*(x-k))_k$ of $(g(x-k))_k$ is given by:

$$\hat{g}^*(\xi) = \hat{g}(\xi)/\sigma(\xi) \tag{2.7}$$

where

$$\sigma(\xi) = \sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi + 2k\pi)\right|^2.$$

Properties 2.1

i) The dual basis of $(g^*(x-k))_k$ is $(g(x-k))_k$.

ii) The functions f of V_0 are written as,

$$f(x) = \sum_{k\in\mathbb{Z}} \alpha_k g(x-k) \tag{2.8}$$

where

$$\left(\sum_{k\in\mathbb{Z}} |\alpha_k|^2\right) \simeq \|f\|_2 .$$

iii) Using the Fourier transform, we get:

$$\hat{f}(\xi) = M(\xi)\hat{g}(\xi)$$

where

$$M(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi} \in L^2(0, 2\pi)$$

and

$$M(\xi + 2\pi) = M(\xi)$$

iv) If $(\varphi(x-k))_{(k\in\mathbb{Z})}$ is an other Riesz basis of V_0, then we get

$$\hat{\varphi}(\xi) = m(\xi)\hat{g}(\xi)$$

where

$$m(\xi + 2\pi) = m(\xi)$$

and

$$0 < \gamma_1 \le |m(\xi)| \le C_1.$$

v) It is obvious to construct an orthonormal basis $(\varphi(x-k))_{(k\in\mathbb{Z})}$ of V_0. We must have two properties:

a) $\varphi(x-k)$ is a Riesz basis and the dual basis of $\varphi(x-k)$ must be $\varphi(x-k)$.

b) We must get

$$\hat{\varphi}(\xi) = m(\xi)\hat{g}(\xi),$$

and

$$\sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(\xi + 2k\pi)\right|^1 = 1.$$

This sequence can be expressed as $|\mu(\xi)|^2\ \sigma(\xi)$ and then we get

$$\left|m(\xi)\right|^2 = \frac{1}{\sigma(\xi)}.$$

It is possible due to the inequality (2.5).

Remark 2.2 We deduce immediately the following properties of g:

i) The functions $(g(x-k))_{(k\in\mathbb{Z})}$, form a Riesz basis of V_0, if we have:

$$0 \le \gamma \le \sum \left|\hat{g}(\xi + 2k\pi)\right|^2 \le C. \tag{2.9}$$

ii) The $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$; if the function g veries

$$\hat{g}(0) \ne 0 \ \text{ and } \ \hat{g}(2k\pi) = 0, k \in \mathbb{Z}^*. \tag{2.10}$$

iii) The inclusion $V_o \subset V_1$ is true when

$$g(x) \subset V_1$$

or

$$g(x) = \sum_{k\in\mathbb{Z}} \alpha_k g(2x + k) \tag{2.11}$$

where

$$\alpha_k = \int_{-\infty}^{\infty} g(x) g^*(2x + k)dx, \text{ and } \alpha_k \in \ell^2$$

iv) The dual basis in V_1 of the functions $(g(2x-k))_{(k\in\mathbb{Z})}$ is given by

$$(2g^*(2x + k))_{(k\in\mathbb{Z})}$$

v) If $g(x)$ is regular, then the function

$$\sigma(\xi) = \sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi + 2k\pi)\right|^2$$

is regular and 2π-periodic. The function $\frac{1}{\sigma(\xi)}$ have the same properties and can be written as:

$$\frac{1}{\sigma(\xi)} = \sum_{k\in\mathbb{Z}} \gamma_k e^{ik\xi}$$

where the coefficients γ_k are rapidly decreasing, then the dual function is dened by

$$g^*(x) = \sum_{k\in\mathbb{Z}} \gamma_k g(x + k).$$

We conclude that the function $g^*(x)$ is regular and the coefficients α_k are rapidly decreasing

vi) Using the Fourier transform in (2.11), we get:

$$\hat{g}(\xi) = \omega(\xi/2)\hat{g}(\xi/2) \tag{2.12}$$

where

$$\omega(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{ik\xi}.$$

3. Biorthogonal Multiresolution Analyses

Definition 3.1 Let A, B two closed subspaces of a Hilbert space H. A and B are in duality for the scalar product of H if

$$H = A \oplus B^{\perp}.$$

We have then the following result.

Proposition 3.1 Let A, B be two closed subspaces of a Hilbert space H. Then, the following properties are equivalent:

i) $H = A \oplus B^{\perp}$.

ii) The orthogonal projector from H on B is an isomorphism from A to B.

iii) For every Riesz basis (b_α) of B there exists a Riesz basis (a_α) of A such that

$$< a_\alpha / b_\beta >_H = \delta_{\alpha,\beta}.$$

iv) There exists a continuous projector P such that

$$P(H) = A \text{ and } P^{-1}\{0\} = B^{\perp}.$$

Proof. In fact, we define P and P^* by

$$P(h) = \sum_{\alpha} < h / b_\alpha >_H a_\alpha,$$

$$P^*(h) = \sum_{\alpha} < h / a_\alpha >_H b_\alpha.$$

Then the conditions described above are symmetric for A and B.

Definition 3.2 A multiresolution analysis is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear sub-spaces of $L^2(\mathbb{R})$ such that:

(i) $V_j \subset V_{j+1}$.

(ii) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$.

(iii) $\cap V_j = \{0\}$ and $\cup V_j$ is dense in $L^2(\mathbb{R})$.

(iv) $f(x) \in V_o \Leftrightarrow f(x-k) \in V_o$, for every $k \in \mathbb{Z}$.

(v) There exists a function $g(x)$ in V_o such that the sequence $\{g(x-k)\}_{k\in\mathbb{Z}}$ form a Riesz basis of V_o:

Properties 3.1

(i) If we denote P_j the orthogonal projector from $L^2(\mathbb{R})$ in V_j. Then the property (iii) of Definition 3.2 can be expressed by the projectors P_j. We have

$$\lim_{j\to-\infty}\|P_jf\|_2 = 0 \text{ and } \lim_{j\to-\infty}\|f-P_jf\|_2 = 0. \tag{3.1}$$

(ii) The properties (ii) and (iv) give

$$P_jf = \sum_{k\in\mathbb{Z}} C_{j,k}g_{j,k} \text{ where } g_{j,k}(x) = g(2^jx-k).$$

Definition 3.3 Let V_j and V_j^* be two multiresolution analyses of $L^2(\mathbb{R})$. Then $(Vj,\ V_j^*)$ is called a biorthogonal multiresolution analysis of $L^2(\mathbb{R})$ if

$$L^2(\mathbb{R}) = V_0 \oplus (V_0^*)^{\perp}.$$

Remark 3.1

i) It is clear that we have, for every $j \in \mathbb{Z}$, the equality

$$L^2(\mathbb{R}) = V_2 \oplus (V_j^*)^{\perp}.$$

ii) If we consider the application $f \to P_jf$ where P_j is the projector from $L^2(\mathbb{R})$ into V_j parallel to $(V_j^*)^{\perp}$. In this case, we say that we have a biorthogonal multiresolution analysis denoted by BMRA.

For a BMRA, we have the following properties:

a)

$$P_joP_{j+1} = P_{j+1}oP_j = P_j. \tag{3.2}$$

b)

$$\lim_{j\to-\infty}\|P_jf\|_2 = 0 \text{ and } \lim_{j\to-\infty}\|f-P_jf\|_2 = 0. \tag{3.3}$$

for $f \in L^2(\mathbb{R})$.

c) $(g(x-k))_{k\in\mathbb{Z}}$ and $(g^*(x-k))_{k\in\mathbb{Z}}$ are respectively the Riesz basis of V_0 and V_0^* such that

$$< g(x-k)/g^*(x-l) >= \delta_{k,l}, \tag{3.4}$$

and we have

$$P_jf = \sum_{k\in\mathbb{Z}} < f/g_{j,k}^* > g_{j,k},$$

where

$$g_{j,k}(x) = 2^{\frac{j}{2}}g(2^jx-k),$$

$$g^*_{j,k}(x) = 2^{\frac{j}{2}} g^*(2^j x - k).$$

d) $(x-k))_{k\in\mathbb{Z}}$ and $(\gamma^*(x-k))_{k\in\mathbb{Z}}$ are respectively the Riesz basis of $W_0 = V_1 \cap (V_0^*)^\perp$ and $W_0^* = V_1^* \cap (V_0)^\perp$ such that

$$< \gamma(x-k)/\gamma^*(x-l) >= \delta_{k,l'} \tag{3.5}$$

The projector Q_j from $L^2(\mathbb{R})$ on $W_j = V_{j+1} \cap (V_j)^\perp$ parallel to $(W_j^*)^\perp$ is given by $Q_j = P_{j+1} - P_j$ and we have

$$Q_j f = \sum_{k\in\mathbb{Z}} < f/\gamma_{j,k} > \gamma_{j,k},$$

where

$$\gamma_{j,k}(x) = 2^{\frac{j}{2}} \gamma(2^j x - k),$$

$$\gamma^*_{j,k}(x) = 2^{\frac{j}{2}} \gamma^*(2^j x - k).$$

The functions g and g^* are called conjugate scaling functions of the biorthogonal multiresolution analysis (V_j, V_j^*) and the functions γ and γ^* are called the associated wavelets.

The functions g, g^*, γ and γ^* have to be compactly supported in order to get fast wavelet transformations. If we assume that supp $g = [N_1, N_2]$ and supp $g^* = [N_1^*, N_2^*]$, then we have the scale equations.

$$g\left(\frac{x}{2}\right) = \sum_{k=N_1}^{N_2} a_k g(x-k). \tag{3.6}$$

$$g^*\left(\frac{x}{2}\right) = \sum_{k=N_1}^{N_2} a_k g^*(x-k). \tag{3.7}$$

$$\hat{g}(2\xi) = M_0(e^{-i\xi})\hat{g}(\xi), \tag{3.8}$$

where

$$M_0(z) = \frac{1}{2}\sum_{k=N_1}^{N_2} a_k z^k.$$

$$\hat{g} * (2\xi) = M_0^*(e^{-i\xi})\hat{g} * (\xi), \tag{3.9}$$

where

$$M_0^*(z) = \frac{1}{2}\sum_{k=N_1^*}^{N_2^*} a_k z^k.$$

$$\hat{\gamma}(2\xi) = e^{-i(2N-1)\xi} M_0^*(-e^{i\xi})\hat{g}(\xi). \tag{3.10}$$

$$\hat{\gamma} * (2\xi) = e^{-i(2N-1)\xi} M_0(-e^{i\xi})\hat{g} * (\xi). \tag{3.11}$$

Then, we obtain

$$\sup\gamma = \left[\frac{1-N_2^*+N_1}{2}, \frac{1+N_2^*-N_1^*}{2}\right]. \tag{3.12}$$

$$\sup\gamma^* = \left[\frac{1-N_2+N_1^*}{2}, \frac{1+N_2^{**}-N_1}{2}\right]. \tag{3.13}$$

If we assume that $g \in H^1$ then there exist two conjugate scaling functions G, G such that

$$g'(x) = G(x) - G(x-1), \tag{3.14}$$

and

$$(G^*(x))' = g^*(x+1) - g^*(x). \tag{3.15}$$

Moreover, if we denote by P_j (resp $P_j^{(1)}$ associated with (G, G^*)) the projector from $L^2(\mathbb{R})$ into V_j (resp $V_j^{(1)}$ parallel to $(V_j^*)^\perp$ (resp $(V_j^{*(1)}))^\perp$, then we have the following commutation property

$$\frac{d}{dx} oP_j = P_j^{(1)} o \frac{d}{dx}. \tag{3.16}$$

This property provides essentially from (3.15). Moreover, the associated functions M_1, M_1^*, the wavelets γ_1^* and the projectors $Q_j^{(1)}$, $Q_j^{*(1)}$ satisfy

$$M_1(z) = \frac{2}{1+z} M_0(z). \tag{3.17}$$

$$M_1(z) = \frac{1+z}{2z} M_0^*(z). \tag{3.18}$$

$$\gamma' = 4\gamma_1. \tag{3.19}$$

$$\gamma_1^{*\prime} = -4\gamma^*. \tag{3.20}$$

$$\frac{d}{dx} o Q_j = Q_j^{(1)} o \frac{d}{dx}. \tag{3.21}$$

This property is adapted for the study of divergence free vectors Lemarié-Rieusset (1992).

4. The Study of Regular Spaces Functions

We use the integration and derivation method described above for constructing a biorthogonal multiresolution analysis. As applications, we prove that these analyses are adapted to study regular functions.

Theorem 4.1 Let V_j be the orthogonal multiresolution analysis of $L^2(\mathbb{R})$ with the scaling function φ of class $C^m(m \in \mathbb{N}^*)$. We denote by $V_j^{(m)}$ and $V_j^{*(m)}$ the multiresolution analysis constructed by m derivations and m integrations. Then $V_j^{(m)}$ and $V_j^{*(m)}$ form a biorthogonal multiresolution analysis of $L^2(\mathbb{R})$. Moreover, if we denote by $P_j^{(m)}$ the projector on $V_J^{(m)}$ parallel to $[V_j^{*(m)}]^{\perp}$, we have

$$\frac{d}{dx} o P_j^m = P_j^{(m+1)} o \frac{d}{dx}. \tag{4.1}$$

The above-described method can be applied for Riesz bases constrcution of the spaces $V_j^{(m)}$ and $V_j^{*(m)}$. In fact, we define g and g^* by

$$(1 - e^{-i\xi})^m \hat{g}(\xi) = (i\xi)^m \hat{\varphi}(\xi), \tag{4.2}$$

$$(i\xi)^m \hat{g}^*(\xi) = (e^{i\xi} - 1)^m \hat{\varphi}(\xi). \tag{4.3}$$

Proposition 4.1 Let $P_j^{(m)}$ be the projector on $V_j^{(m)}$ parallel to $(V_j^{*(m)})^{\perp}$ and $P_j^{*(m)}$ its adjoint. We dene

$$Q_j^{*(m)} = P_j^{*(m+1)} - P_j^{(m)}$$
$$Q_j^{*(m)} = P_{j+1}^{*(m)} - P_j^{*(m)}$$

Then we have the following commutation properties

$$\frac{d}{dx}(P_j^{(m)} f) = P_j^{(m+1)}\left(\frac{df}{dx}\right) \text{ if } f \in H^1(\mathbb{R}), \tag{4.4}$$

$$\frac{d}{dx}(P^{*(m+1)} f) = P_j^{*(m)}\left(\frac{df}{dx}\right) \text{ if } f \in H^1(\mathbb{R}). \tag{4.5}$$

Proof. To prove this Proposition, it is enough to remark that if $f \in H^1$ and $g \in H^1$, then we have

$$(P_j f, g)_{L^2} = \langle f, P_j^* g \rangle$$

$$\left\langle \frac{df}{dx}, g \right\rangle = -\left\langle f, \frac{dg}{dx} \right\rangle.$$

We can now establish the main result of this section.

Theorem 4.2 Assume that φ is a $C^{p+\varepsilon}$-function, $p \in N^*$, $p \geq m$ and $\varepsilon > 0$. Then we have

i) For $f \in L^2$, $\|f\|_2 \approx \left\|P_0^{(m)} f\right\|_2 + \left(\sum_{j\geq 0} \left\|Q_j^{(m)} f\right\|_2^2\right)^{\frac{1}{2}}$.

ii) For $f \in L^2$, $\|f\|_2 \approx \left\|P_0^{*(m)} f\right\|_2 + \left(\sum_{j\geq 0} \left\|Q_j^{*(m)} f\right\|_2^2\right)^{\frac{1}{2}}$.

iii) For $s \in \mathbb{Z}$ such that $-m \leq s \leq p - m$; we have

- $(f \in H^s) \Leftrightarrow (P_0^{(m)} f \in L^2$ and $\Sigma_{j\geq 0} 4^{js} || Q_j^{(m)} f ||_2^2 < +\infty)$.
- $(f \in H^{-s}) \Leftrightarrow (P_0^{*(m)} f \in L^2$ and $\Sigma_{j\geq 0} 4^{js} || Q_j^{*(m)} f ||_2^2 < +\infty)$.

Proof. The proof of this Theorem is classical in the wavelet theory. The direct inequalities in i) and ii) can be easily obtained from the vaguelet Lemma Jouini et Al (1992) and the inverse inequalities can be obtained by duality. The equivalences in iii) are immediate because if $f \in H^s$ then its norm is equal to $||f||_2 + ||f^{(s)}||_2$. We set

$$f = P_0^{(m)} f + \sum_{j=0}^{\infty} Q_j^{(m)} f,$$

then, we have

$$\|f\|_2 \approx \left\|P_0^{(m)} f\right\|_2 + \left(\sum_{j=0}^{\infty} \left\|Q_j^{(m)} f\right\|_2^2\right)^{frac12},$$

$$f^{(s)} = \left(\frac{d}{dx}\right)^s \left(P_0^{(m)} f + \sum_{j=0}^{\infty} Q_j^{(m)} f\right) = P_0^{(m+s)} f^{(s)} + \sum_{j=0}^{\infty} Q_j^{(m+s)} f^{(s)},$$

$$\left\|f^{(s)}\right\|_2 \approx \left\|p_0^{(m+s)}{}_0 f^{(k)}\right\|_2 + \left(\sum_{j=0}^{\infty} \left\|Q_j^{(m+s)} f^{(s)}\right\|_2^2\right)^{\frac{1}{2}}.$$

thus, we obtain

$$\left\|P_0^{(m+s)} f^{(s)}\right\|_2 = \left\|\left(\frac{d}{dx}\right)^s (P_0^{(m)} f)\right\|_2 \leq C \left\|P_0^{(m)} f\right\|_2,$$

$$\left\|Q_0^{(m+s)} f^{(s)}\right\|_2 = \left\|\left(\frac{d}{dx}\right)^s (Q_0^{(m)} f)\right\|_2 \approx 2^{js} \left\|Q_0^{(m)} f\right\|_2.$$

Then the characterization of H^s is immediate.

Remark 4.1

i) We can use the method described in this paper to construct in an elementary way general biorthogonal multiresolution analyses in bounded domains as those involved in (Ajmi et Al (2001), Bibi et Al (2008), Jouini (2007) and Jouini et Al (2013)).

ii) Biorthogonal multiresolution analyses have many applications as Numerical Simulation for elliptic problems, digital image processing and potential applications (Berrezoug et Al (2009), Rubeck (2012),and Elk–eet Al (2004)).

5. Conclusion

In this paper, we described more general constructions of biorthogonal multiresolution analyses. More precisely, we constructed biorthogonal systems which are provided by dyadic translations and dilatations of a new mother wavelet. By using the method of derivation and integration, we obtain new regular biorthogonal multiresolution analyses which satisfy the commutation properties (3.16), (3.21) and (4.1). We then deduced that these analyses are well adapted for the study of the Sobolev spaces $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$ ($s \in \mathbb{N}$).

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